NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-80

An Infinite System of Linear Equations Arising in Diffraction Theory

W. MAGNUS

CONTRACT NO. AF-19(122)-42

JUNE 1955

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. EM-80

AN INFINITE SYSTEM OF LINEAR EQUATIONS ARISING IN DIFFRACTION THEORY

W. Magnus

W. Magnus

W. Magnus

Morris Kline

Morris Kline
Project Director

The research reported in this document has been made possible through support and sponsorship extended by the Air Force Cambridge Research Center, under Contract No. AF-19(122)-42. It is published for technical information and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1955

Abstract

An infinite system of linear equations is investigated which arises in the theory of diffraction by a circular aperture. The coefficients of this system depend on a parameter α. It is proved that the solution of the system can be found to any degree of approximation by solving the first N equations for the first N unknowns, for N sufficiently large, if α is either real or purely imaginary.

Table of Contents

		Page
1.	Introduction	1
2.	A Fredholm integral equation equivalent to Bouwkamp's system of linear equations	3
3.	Various forms of the kernel $G(s,\sigma,\alpha)$	6
4.	Existence of the solution	9
5.	Approximate solution of the set of linear equations	13
	References	19

1. Introduction

We consider the problem of diffraction of a plane scalar wave normally incident on a plane screen with a circular aperture. Let u be the solution of

$$\Delta u + K^2 u = 0$$

which represents the diffracted wave and let ρ, Θ be polar coordinates on the screen such that $\rho = 0$ represents the center of the aperture. We assume that on the screen $u = 0$. Let the radius of the aperture be a and let $\Phi(\rho)$ be the value of u in the aperture, $0 \leq \rho \leq a$. Then the following result was derived by Bouwkamp [1] using the variational method of Levine and Schwinger [2]:

If $\Phi(\rho)$ is expanded in a series of Legendre polynomials of the type

$$\Phi(\rho) = \sum_{n=0}^{\infty} b_n P_{2n+1} \left(\sqrt{1-\rho^2/a^2}\right) \tag{1.1}$$

then the b_n satisfy the system of infinitely many linear equations

$$\sum_{n=0}^{\infty} d_{m,n} b_n = \frac{6}{i\alpha} \delta_{m,0}, \tag{1.2}$$

where $\alpha = Ka$, $\delta_{0,0} = 1$ and $\delta_{m,0} = 0$ if $m \neq 0$, and where

$$d_{m,n} = \left(\frac{6}{\alpha}\right)^2 \frac{\Gamma(n+3/2)}{n!} \frac{\Gamma(m+3/2)}{m!} g^*_{m,n}(\alpha) \tag{1.3}$$

and

$$g^*_{m,n}(\alpha) = \int_0^{\infty} \frac{\sqrt{v^2-1}}{v^2} J_{2m+3/2}(\alpha v) J_{2n+3/2}(\alpha v)\, dv. \tag{1.4}$$

In Eq. (1.4), J denotes the Bessel function of the first kind, and

$$\sqrt{v^2-1} = -i\sqrt{1-v^2}, \quad \sqrt{1-v^2} > 0 \tag{1.5}$$

for $0 \leq v < 1$. The diffracted amplitude at infinity in the forward direction is given by

$$(1.6) \qquad A_1 = -\frac{i\alpha}{3} b_o .$$

The occurrence of an infinite system of linear equations is typical for the variational method. The system (1.2) has the advantage that its matrix reduces to a diagonal matrix if $\alpha \to 0$ (see Eq. (2.9) below). It has other advantages for $\alpha \to \infty$: it can be shown that the limit

$$\lim_{\alpha \to \infty} \alpha^{-1} g^*_{m,n}(\alpha) = g^{(\infty)}_{m,n}$$

exists and that the infinite matrix with the general element $g^{(\infty)}_{m,n}$ has a formal inverse. But we shall not use this fact here.

The main purpose of the present report is to establish that Bouwkamp's system (1.2) of infinitely many linear equations can be used effectively for the purpose of computing the field in the aperture and the transmission coefficient A_1 of (1.6). It is shown that this can be done approximately for <u>all real</u> values of α by solving the first N+1 equations (1.2) for the first N+1 unknowns (assuming that the remaining unknowns have the value zero). For sufficiently small values of $|\alpha|$, this has already been proved for Levine and Schwinger's system of linear equations in an earlier report [3]. The present approach also gives some general insight into the dependency of the unknowns b_n on the parameter α; in particular, we show that the b_m are analytic functions of the complex variable α which do not have any singularities on the imaginary α-axis (see Theorem I).

2. A Fredholm integral equation equivalent to Bouwkamp's system of linear equations

Instead of Bouwkamp's coefficients b_n for the expansion of the field in the aperture we now introduce the quantities

$$(2.1) \qquad s_n = (-1)^n \frac{\Gamma(n+3/2)}{n!} b_n$$

and consider the system of linear equations

$$(2.2) \qquad \sum_{n=0}^{\infty} g_{m,n} s_n = t_m \qquad (n,m = 0, 1, 2, \ldots),$$

where

$$(2.3) \qquad g_{n,m} = g_{n,m}(\alpha) = (-1)^{n+m} \int_0^{\infty} \frac{\sqrt{v^2-1}}{v} J_{2m+3/2}(\alpha v)\, J_{2n+3/2}(\alpha v) \frac{dv}{v} .$$

If we put $\delta_{m,n} = 0$ $(m \neq n)$, $\delta_{n,n} = 1$, and define

$$(2.4) \qquad t_m = -\frac{i\alpha}{6} \frac{m!(-1)^m}{\Gamma(m+3/2)} \delta_{m,0},$$

then the system (2.2) is the equivalent of Bouwkamp's system for the new unknows s_n and the new right-hand sides t_m. The conditions for the b_n imply the following conditions for the s_n: first, that

$$(2.5) \qquad \lim_{\varepsilon \to 0} \sum_{0}^{\infty} \frac{(-1)^n n!}{\Gamma(n+3/2)} s_n P_{2n+1}(1-\varepsilon)$$

exists, and second, that

$$(2.6) \qquad \sum_{n=0}^{\infty} \left| \frac{s_n}{n+\frac{1}{2}} \right|^2 < \infty$$

Condition (2.6) can be derived from the fact that the field in the aperture must be square integrable and from the equation

$$\lim_{n\to\infty} \frac{\Gamma(n+3/2)}{n!\,\sqrt{n+1/2}} = 1.$$

We shall now investigate the system (2.2) under the assumption that

$$(2.7)\qquad \sum_{m=0}^{\infty} (4m+3)\,|t_m| < \infty,$$

which is obviously justified for the particular values of the t_m in (2.4). For this purpose we need the following

<u>Lemma 1.</u> Let

$$(2.8)\qquad G \equiv G(s,\sigma,\alpha) = \int_0^{\infty} \frac{\sqrt{v^2-1} - v}{v} \sin(\alpha v s)\sin(\alpha v \sigma)\,dv,$$

where

$$\sqrt{v^2-1} = -i\sqrt{1-v^2} \quad \text{for} \quad 0 \le v \le 1 .$$

Then

$$(2.9)\qquad \int_0^1\int_0^1 G(s,\sigma,\alpha)\,P_{2n+1}(s)\,P_{2m+1}(\sigma)\,ds\,d\sigma$$

$$= \frac{\pi}{2\alpha}\left[g_{m,n}(\alpha) - \delta_{m,n}/(4n+3)\right] .$$

<u>Proof:</u> From the 'degenerate' addition theorem for Bessel functions [see [6], p. 102] we have

$$(2.10)\qquad \sin(\alpha v s) = [\pi/(2\alpha)]^{1/2} \sum_{m=0}^{\infty} (-1)^m (4m+3)\, J_{2m+3/2}(\alpha v)\, P_{2m+1}(s).$$

The series in (2.10) converges absolutely and uniformly for all real values of α, v and s if $0 \le s \le 1$.

By substituting the right-hand side of (2.10) for $\sin(\alpha vs)$ [and the corresponding expression for $\sin(\alpha v\sigma)$] in the right-hand side of (2.8) we find

$$(2.11)\quad G = \frac{\pi}{2\alpha}\int_0^\infty \frac{\sqrt{v^2-1}-v}{v^2}\,dv \sum_{m,n=0}^{\infty} (-1)^{n+m}(4m+3)(4n+3)\; J_{2m+(3/2)}(\alpha v)\; J_{2n+(3/2)}(\alpha v) \cdot P_{2m+1}(s)\, P_{2n+1}(\sigma)\,.$$

Now we can derive (2.9), with the $g_{n,m}$ as defined in (2.3), from (2.11); for this we use the orthogonality relations

$$(2.12)\qquad \int_0^1 P_{2m+1}(s)\, P_{2n+1}(s)\, ds = \frac{\delta_{m,n}}{4m+3}\;,$$

and the following special case of the Sonine-Schafheitlin integral (see [6], p. 51):

$$(2.13)\qquad \int_0^\infty J_{2m+3/2}(\alpha v)\; J_{2n+3/2}(\alpha v)\,\frac{dv}{v} = \frac{\delta_{m,n}}{4m+3}\;.$$

This derivation of (2.9) requires exchanging the order of summation and integration in (2.11), which is easily justified.

Lemma 1 leads immediately to an integral equation of the second kind whose kernel is $\frac{2\alpha}{\pi}\,G(s,\sigma,\alpha)$; this equation is equivalent to the system (2.2). Indeed, if we introduce the functions

$$(2.14)\qquad S(\sigma) = \sum_{n=0}^{\infty} s_n\, P_{2n+1}(\sigma),$$

$$(2.15)\qquad T(s) = \sum_{m=0}^{\infty} t_m\, P_{2m+1}(s),$$

then Eqs. (2.9), (2.2) and (2.12) immediately give the result

$$T(s) = S(s) + \frac{2\alpha}{\pi} \int_0^1 G(s,\sigma;\alpha)\, S(\sigma)\, d\sigma. \tag{2.16}$$

It is easy to see that any continuous solution $S(s)$ of (2.16) leads, via (2.14), (2.15) and (2.9), to a solution of (2.2) satisfying (2.5) and (2.6). This is due to the fact that $G(s,\sigma,\alpha)$ can be expanded in a power series in $s + \sigma$ and $|s-\sigma|$ which is everywhere convergent, (see (3.2), (3.3) below). Therefore, the integral on the right-hand side of (2.16) satisfies a Lipschitz condition if $S(\sigma)$ is continuous. From this it follows that $S(s) - T(s)$ can be expanded in a series of type (2.15) satisfying (2.5) and (2.6), and from (2.7) the same follows for $S(s)$ itself.

It is more difficult to show that every admissible solution of (2.2) leads to a solution $S(s)$ of (2.16). We can omit a discussion of this problem because we shall prove later that (2.16) has a continuous solution $S(\sigma)$ for all finite real positive values of α. Therefore, we always have an admissible solution of (2.2), and it follows from well-known uniqueness theorems (Bouwkamp[4], Meixner[5]) that there cannot be a second such solution. Finally, we shall prove that the solution $S(\sigma)$ obtained from (2.16) could also be derived from (2.2) by solving the N first equations for the N first unknowns in (2.2) and letting $N \to \infty$ (see Section 5).

3. Various forms of the kernel $G(s,\sigma,\alpha)$

For a detailed study of the integral equation (2.16) we need various forms of the kernel G. We have:

Lemma 2. The kernel $G(s,\sigma;\alpha)$ can be defined for all complex values of α and for all real values of s,σ by

$$(3.1)\quad G(s,\sigma;\alpha) = \int_0^\infty \left(\sqrt{v^2-1} - v\right) \sin(\alpha vs)\sin(\alpha v\sigma)\, \frac{dv}{v}$$

$$(3.2)\quad = -\frac{i}{2} \int_0^1 \sqrt{1-w^2} \left\{ e^{i\alpha w|s-\sigma|} - e^{i\alpha w(s+\sigma)} \right\} \frac{dw}{w}$$

$$(3.3)\quad = -\frac{\pi}{4} \int_{|s-\sigma|}^{|\sigma+s|} \left\{ J_1(\alpha\tau) + iH_1(\alpha\tau) \right\} \frac{d\tau}{\tau},$$

where J_1 denotes the Bessel function of the first kind and H_1 denotes the Struve function, both of order one. Eqs. (3.2) and (3.3) provide formulas for the analytic continuation of $G(s,\sigma;\alpha)$ into the complex α-plane.

Proof: We shall derive (3.2) for $s \geq \sigma$, that is, for $|s-\sigma| = s-\sigma$. Since G is symmetric in s and σ, this will prove (3.2) completely. By using the elementary formula

$$(3.4)\quad 4\sin(\alpha vs)\sin(\alpha v\sigma) = e^{i\alpha v(s-\sigma)} - e^{i\alpha v(s+\sigma)} + e^{-i\alpha v(s-\sigma)} - e^{-i\alpha v(s+\sigma)},$$

we can split the integral in (3.1) into the two parts:

$$(3.5)\quad G_+ = \int_0^\infty \left(\sqrt{v^2-1} - v\right) \left\{ e^{i\alpha v(s-\sigma)} - e^{i\alpha u(s+\sigma)} \right\} \frac{dv}{v}$$

and

$$(3.6)\quad G_- = \int_0^\infty \left(\sqrt{u^2-1} - v\right) \left\{ e^{-i\alpha u(s-\sigma)} - e^{-i\alpha v(s+\sigma)} \right\} \frac{dv}{v}.$$

By deforming the paths of integration in the complex v-plane, we can write (see Fig. 1):

$$(3.7)\quad G_+ = \int_0^{(1+)} + \int_0^{i\infty}, \qquad G_- = \int_0^{-i\infty}.$$

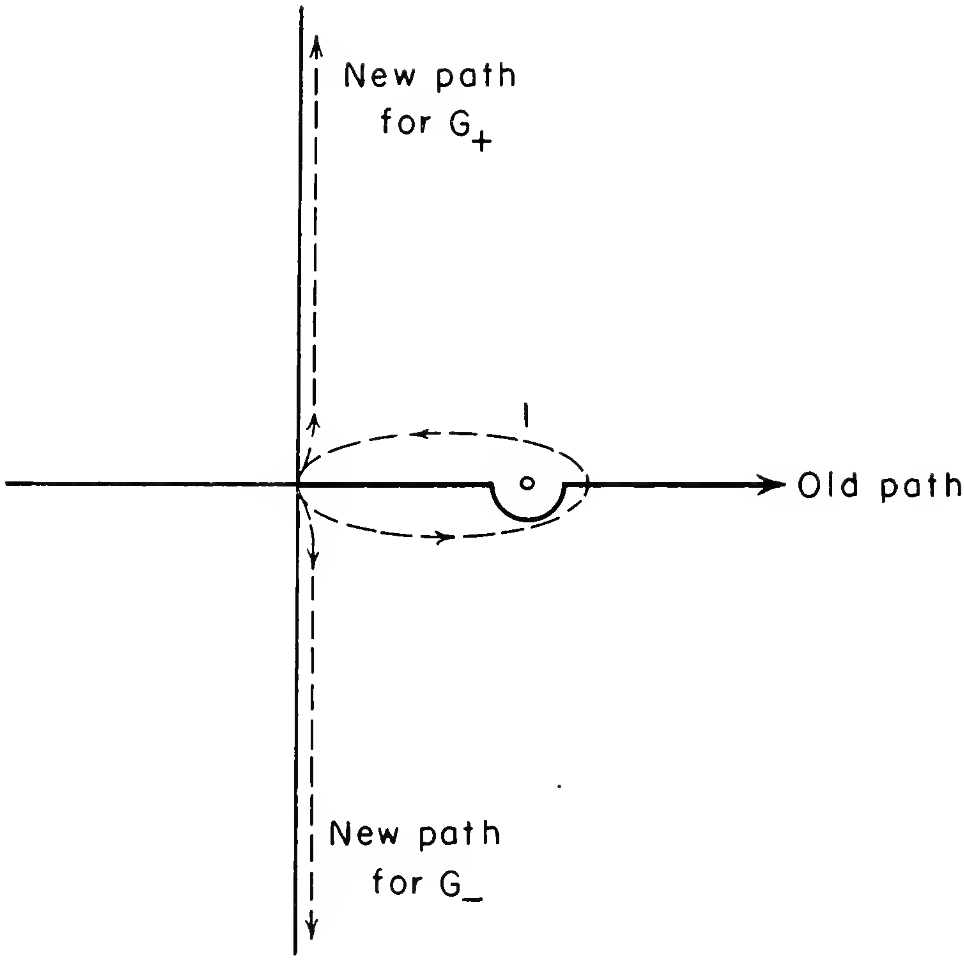

Figure 1

If we take into account that $\sqrt{v^2-1} = -i\sqrt{1-v^2}$ for $0 \leq 0 < 1$ on the old path of integration, we can derive (3.2) from (3.7) by a straightforward computation.

In order to derive (3.3), we observe that (3.2) can be written in the form

$$G(s,\sigma;a) = -\frac{a}{2}\int_0^1 \sqrt{1-w^2}\left\{\int_{|s-\sigma|}^{s+\sigma} e^{iawt}\,dt\right\}dw. \tag{3.8}$$

By exchanging the order of integration in (3.8) and using the integral representations (see Erdélyi[6], pp. 14, 37)

$$(3.9) \qquad \tau^{-1} J_1(\alpha\tau) = \frac{2\alpha}{\pi} \int_0^1 \cos(\alpha\tau w)\sqrt{1-w^2}\,dw$$

$$(3.10) \qquad \tau^{-1} H_1(\alpha\tau) = \frac{2\alpha}{\pi} \int_0^1 \sin(\alpha\tau w)\sqrt{1-w^2}\,dw$$

we find that Eq. (3.3) follows immediately from (3.2).

We shall mention here a result which describes the behavior of G for $\alpha \to \infty$ and which is important in connection with the asymptotic behavior of the field in the aperture for large values of α. We have:

$$(3.11) \qquad \lim_{\alpha \to \infty} G(s,\sigma,\alpha) = -\frac{i}{2}\log\left|\frac{s+\sigma}{s-\sigma}\right| .$$

The proof will not be given here.

4. Existence of the solution

In this section, the following result will be proved:

Theorem I. The Fredholm integral equation

$$(4.1) \qquad T(s) = S(s) + \frac{2\alpha}{\pi}\int_0^1 G(s,\sigma;\alpha)\, S(\sigma)\, d\sigma$$

has exactly one continuous solution $S(\sigma)$ if $T(s)$ satisfies (2.7) and if one of the following conditions is satisfied:

(i) α is real
(ii) α is purely imaginary
(iii) $|\alpha|$ is sufficiently small.

For the proof of Theorem I we shall use the representation (3.2) of the kernel G. For the case where (iii) is satisfied, Theorem I has been proved elsewhere[3]. In fact, Eq. (4.1) is equivalent to the infinite system

of linear equations treated in [3], whose solution was shown to be an analytic function of α for sufficiently small values of $|\alpha|$.

In order to prove Theorem I for the case where (i) is satisfied, we may proceed as follows:

Eq. (4.1) is a Fredholm integral equation with a continuous kernel (according to (3.2)). It is known (see [7]) to have a uniquely determined continuous solution $S(\sigma)$ for any $T(s)$, provided that the homogeneous equation

$$(4.2) \qquad S(s) + \frac{2\alpha}{\pi} \int_0^1 G(s,\sigma;\alpha)\, S(\sigma)\, d\sigma = 0$$

does not have a continuous solution different from zero. Now let α be real, and put

$$(4.3) \qquad S(s) = u(s) + i\, v(s),$$

$$(4.4) \qquad \frac{2\alpha}{\pi}\, G(s,\sigma;\alpha) = K(s,\sigma) + i\, L(s,\sigma),$$

where u, v, K, L are real-valued functions. Then, by splitting (4.2) into its real and imaginary parts, we find

$$(4.5) \qquad \int_0^1 \left[K(s,\sigma)\, u(\sigma) - L(s,\sigma)\, v(\sigma)\right] d\sigma + u(s) = 0,$$

$$(4.6) \qquad \int_0^1 \left[K(s,\sigma)\, v(\sigma) + L(s,\sigma)\, u(\sigma)\right] d\sigma + v(s) = 0.$$

If we multiply Eq. (4.5) by $-v(s)$ and Eq. (4.6) by $u(s)$ we find, after adding the resulting equations and integrating with respect to s:

$$(4.7) \qquad \int_0^1 \int_0^1 \left[v(s)\, L(s,\sigma)\, v(\sigma) + u(s)\, L(s,\sigma)\, u(\sigma)\right] ds\, d\sigma = 0.$$

If we can show that $L(s,\sigma)$ is a negative definite kernel for $\alpha > 0$, that is, if

$$(4.8) \qquad \int_0^1 \int_0^1 v(s)\, L(s,\sigma)\, v(\sigma)\, ds\, d\sigma < 0$$

whenever the continuous function $v(s)$ does not vanish identically, then we know that (4.7) can be satisfied only if $u(s) = v(s) = S(s) \equiv 0$. Therefore, Part (i) of Theorem I would be proved if (4.8) were proved. Now we have from (3.2)

$$(4.9) \qquad L(s,\sigma) = \frac{\alpha}{\pi} \int_0^1 \Big[\cos\,[\alpha w(s+\sigma)] - \cos\,[\alpha w(s-\sigma)]\Big] \sqrt{1-w^2}\; \frac{dw}{w}$$

$$= -\frac{2\alpha}{\pi} \int_0^1 \sin(\alpha ws)\, \sin(\alpha w\sigma)\, \sqrt{1-w^2}\; \frac{dw}{w},$$

and therefore the left-hand side of (4.8) can be written as

$$(4.10) \qquad -\frac{2\alpha}{\pi} \int_0^1 \int_0^1 \int_0^1 \sin(\alpha ws)\, \sin(\alpha w\sigma)\, v(s)\, v(\sigma)\, \sqrt{1-w^2}\; ds\, d\sigma\, \frac{dw}{w}$$

$$= -\frac{2\alpha}{\pi} \int_0^1 \left\{ \int_0^1 \sin(\alpha ws)\, v(s)\, ds \right\}^2 \sqrt{1-w^2}\; \frac{dw}{w}\,.$$

The integrand in the right-hand side of (4.10) is non-negative, and therefore the integral with respect to w can vanish only if

$$(4.11) \qquad \int_0^1 \sin(\alpha ws)\, v(s)\, ds = 0$$

for $0 \leq w \leq 1$. Now the left-hand side of (4.11) is an analytic function of w. Expanding it into a power series in w, and observing that all the coefficients of this series must vanish, we find that

$$(4.12) \qquad \alpha^n \int_0^1 s^n\, v(s)\, ds = 0, \qquad (n = 0, 1, 2, \ldots) \quad .$$

If $\alpha > 0$, it follows from (4.12) that $v(s) \equiv 0$ if $v(s)$ is continuous. If $\alpha = 0$, then $S = u + iv$ vanishes identically because of (4.2). If $\alpha < 0$, then L becomes positive definite, and (4.11) is true again. This completes the proof of assertion (i) of Theorem I.

To prove assertion (ii) of Theorem I we put $\alpha = i\beta$, where β is real. In this case, (4.2) becomes

$$(4.13) \qquad S(s) + \frac{\beta}{\pi} \int_0^1 Q(s,\sigma;\beta)\, S(\sigma)\, d\sigma = 0,$$

where, according to (3.2):

$$(4.14) \qquad Q(s,\sigma;\beta) = \int_0^1 \sqrt{1-w^2} \left[e^{-\beta w|s-\sigma|} - e^{-\beta w(s+\sigma)} \right] \frac{dw}{w} \; .$$

Obviously, (4.13) cannot have a non-trivial continuous solution $S(\sigma)$ if $Q(s,\sigma;\beta)$ is a positive kernel for $\beta > 0$ and a negative kernel for $\beta < 0$. It may suffice to treat the case where $\beta > 0$. Let

$$(4.15) \qquad T(s,\rho) = \begin{cases} e^{\beta\rho w} & \text{if } s > \rho \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$(4.16) \qquad Q(s,\sigma;\beta) = \int_0^1 2\beta e^{-\beta w(s+\sigma)} \left\{ \int_0^1 T(s,\rho)\, T(\sigma,\rho)\, d\rho \right\} \sqrt{1-w^2}\, dw \; .$$

In order to compute

$$(4.17) \qquad \int_0^1 \int_0^1 S(s)\, Q(s,\sigma;\beta)\, S(\sigma)\, ds\, d\sigma,$$

we shall introduce

$$(4.18) \qquad \phi(\rho,w) = \int_0^1 e^{-\beta ws}\, T(s,\rho)\, S(s)\, ds = \int_\rho^1 e^{+\beta w(\rho-s)}\, S(s)\, ds.$$

Obviously, $\phi(\rho,w)$ is a continuous function in ρ and w if $S(s)$ is continuous, and ϕ can vanish identically only if $S(s)$ is identically zero. By combining (4.16) and (4.18) we find that the double integral in (4.17) can be written in the form

$$(4.19) \qquad 2\beta \int_0^1 \int_0^1 [\phi(\rho,w)]^2 \sqrt{1-w^2}\, d\rho\, dw.$$

Since the expression (4.19) is positive for $\beta > 0$ unless ϕ vanishes identically, we see that (4.19) is positive unless $S(s)$ vanishes identically, and this proves assertion (ii) of Theorem I.

5. Approximate solution of the set of linear equations

It was mentioned in the introduction that given an infinite set of linear equations, it is important to know whether an approximate solution can be found by solving the first N+1 equations for the first N+1 unknowns, i.e., by proceeding as if all of the other unknowns were zero. We shall call this procedure 'solving for the N+1 first unknowns', and we shall prove that it is a legitimate method in the case of Bouwkamp's equations for all real values of the parameter α. For this purpose, we shall introduce still another set of unknowns σ_n:

$$(5.1) \qquad \sigma_n = \frac{s_n}{\sqrt{4n+3}} = (-1)^n \frac{\Gamma(n+3/2)}{n!\,\sqrt{4n+3}}\, b_n.$$

Correspondingly, instead of taking the t_m of Eqs. (2.2), we introduce the

$$(5.2) \qquad \tau_m = \frac{t_m}{\sqrt{4m+3}},$$

and instead of the $g_{n,m}$ there we define another set of coefficients

(5.3) $$\gamma_{n,m} = \sqrt{4n+3}\ \sqrt{4m+3}\ g_{n,m}\ .$$

Then we have instead of Eqs. (2.2):

(5.4) $$\sum_{n=0}^{\infty} \gamma_{m,n}\, \sigma_n = \tau_m.$$

According to Lemma 1, (Eq. (2.9)), we may write:

(5.5) $$\gamma_{n,m} = \delta_{m,n} + \frac{2a}{\pi} K_{n,m}\ ,$$

where

$$K_{n,m} = \int_0^1\int_0^1 G(s,\sigma,a)\sqrt{4n+3}\ P_{2n+1}(s)\sqrt{4m+3}\ P_{2m+1}(\sigma)\ ds\ d\sigma.$$

We shall use the notation

(5.6) $$\phi_n(s) = \sqrt{4n+3}\ P_{2n+1}(s),$$

where the functions $\phi_n(s)$ form a complete orthonormal set of functions for the interval $0 \leq s \leq 1$. Therefore we can write

(5.7) $$G(s,\sigma,a) = \sum_{n,m=0}^{\infty} K_{n,m}\, \phi_n(s)\phi_m(\sigma).$$

The function G in (5.7) is approximated by

(5.8) $$G_N(s,\sigma,a) = \sum_{n,m=0}^{N} K_{n,m}\, \phi_n(s)\phi_n(\sigma),$$

in the sense that

(5.9) $$\lim_{N\to\infty} \int_0^1\int_0^1 \left[G(s,\sigma,a) - G_N(s,\sigma,a)\right]^2 ds\ d\sigma = 0$$

for any fixed value of α. Now we can formulate and prove the following result for the solution for the first N+1 unknowns of the set of linear equations (5.4):

Theorem II: The determinant Δ_N of the finite set of linear equations

$$\sum_{n=0}^{N} \gamma_{m,n}\, \sigma_n = \tau_m \qquad (m = 0, 1, \ldots N) \tag{5.10}$$

does not vanish for any real value of α. The unique solution

$$\sigma_n = \sigma_{n,N} \qquad (n = 0, 1, \ldots, N) \tag{5.11}$$

of (5.10) has the property that

$$\lim_{N \to \infty} \sigma_{n,N} = \sigma_{n,\infty} \tag{5.12}$$

exists for all n and that the $\sigma_{n,\infty}$ satisfy the original equations (5.4).

We shall derive Theorem II from the following

Theorem III: For real values of α, the integral equation

$$S_N(s) + \frac{2\alpha}{\pi} \int_0^1 G_N(s,\sigma,\alpha)\, S_N(\sigma)\, d\sigma = T_N(s) \tag{5.13}$$

has exactly one solution

$$S_N(\sigma) = \sum_{n=0}^{N} \sigma_{n,N}\, \phi_n(\sigma) \tag{5.14}$$

if

$$T_N(s) = \sum_{n=0}^{N} \tau_n \phi_n(s). \tag{5.15}$$

<u>Proof:</u> Let $L_N(s,\sigma,\alpha)$ be the imaginary part of $G_N(s,\sigma,\alpha)$. If the homogeneous equation (5.13) has any solution at all it must be of the type (5.14), and we can write

$$(5.16) \qquad S_N(s) = \sum_{n=0}^{N} (u_n + iv_n)\, \phi_n(\sigma) = U_N(\sigma) + iV_N(\sigma)$$

where u_n, v_n, U_N, and V_n are real and where

$$(5.17) \qquad \int_0^1\int_0^1 U_N(s)\, L_N(s,\sigma,\alpha)\, U_N(\sigma)\, ds\, d\sigma + \\ + \int_0^1\int_0^1 V_N(s)\, L_N(s,\sigma,\alpha)\, V_N(\sigma)\, ds\, d\sigma = 0$$

(see Eq. (4.7)). But (5.17) cannot be satisfied unless $u_n = v_n = 0$ for $n = 0, 1, \ldots, N$, since

$$(5.18) \qquad \int_0^1\int_0^1 U_N(s)\, L_N(s,\sigma,\alpha)\, U_N(\sigma)\, ds\, d\sigma \\ = \int_0^1\int_0^1 U_N(s)\, L(s,\sigma,\alpha)\, U_N(\sigma)\, ds\, d\sigma,$$

where L is the imaginary part $G(s,\sigma,\alpha)$. Now we know from the remarks after (4.8) that the right-hand side of (5.18) is always less than 0 for $\alpha > 0$, and vice versa, unless $U_N \equiv 0$. Therefore, (5.13) has at most one solution and it follows from Fredholm's theory that it has exactly one solution.

Now we can prove Theorem II. If we write the only solution of (5.13) in the form (5.14), the coefficients $\sigma_{n,N}$ must satisfy Eq. (5.10). According to Theorem III, these equations cannot have more than one solution, and therefore the determinant of the system does not vanish. This proves the first part of Theorem II. The second part, namely the proof of (5.12), is somewhat tedious;

we shall outline the proof here. $S(s)$ is the uniquely determined solution of (2.16). Then it suffices to show that

$$(5.19) \qquad \lim_{N \to \infty} \int_0^1 |S(s) - S_N(s)|^2 \, ds = 0.$$

In fact, because of Parseval's equation, we have

$$(5.20) \qquad \int_0^1 |S(s) - S_N(s)|^2 \, ds = \sum_{n=0}^{N} \left|\sigma_n - \sigma_{n,N}\right|^2 + \sum_{n=N+1}^{\infty} |\sigma_n|^2,$$

where

$$\sigma_n = \int_0^1 S(s) \, \phi_n(s) \, ds.$$

Therefore, (5.19) implies (5.12). In order to prove (5.19), we may proceed as follows. By subtracting (5.13) from (2.16) we find

$$(5.21) \qquad S(s) - S_N(s) + \frac{2\alpha}{\pi} \int_0^1 G(s,\sigma) \left[S(\sigma) - S_N(\sigma)\right] d\sigma$$

$$= T(s) - T_N(s) - \int_0^1 \left[G(s,\sigma) - G_N(s,\sigma)\right] S_N(\sigma) \, d\sigma.$$

Now let $R(s,\sigma)$ be the resolving kernel of (2.16). According to Theorem I, it follows from Fredholm's theory that R is a continuous function of s and σ such that the solution of (4.1) can be written in the form

$$(5.22) \qquad S(s) = T(s) + \int_0^1 R(s,\sigma) \, T(\sigma) \, d\sigma.$$

If we look upon (5.21) as an integral equation for $S(s) - S_N(s)$, we obtain from an application of (5.22):

$$S(s) - S_N(s) = H_N(s) + \int_0^1 R(s,\sigma)\, H_N(\sigma)\, d\sigma, \tag{5.23}$$

where $H_N(s)$ denotes the right-hand side of (5.21). Since as $N \to \infty$

$$\int_0^1 \left|T(s) - T_N(s)\right|^2 ds \to 0, \tag{5.24}$$

(5.19) is true if

$$\int_0^1 \left| \int_0^1 [G(s,\sigma) - G_N(s,\sigma)]\, S_N(\sigma)\, d\sigma \right|^2 ds \le \varepsilon_N \int_0^1 \left|S_N(\sigma)\right|^2 d\sigma, \tag{5.25}$$

where $\varepsilon_N \to \infty$ as $N \to \infty$. In fact, it follows from (5.25) and (5.23) by a standard procedure that (5.19) holds. Since (5.25) is a consequence of (5.9) and of Schwartz's inequality, Theorem II is true.

References

[1] Bouwkamp, C.J. - Diffraction theory; Reports on Progress in Physics, 17, 35-100 (1954). Also Research Report EM-50, New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research

[2] Levine, H. and Schwinger, J. - On the theory of diffraction by an aperture in a plane screen; Physical Review, 74, 958-974 (1948).

[3] Magnus, W. - Infinite matrices associated with diffraction by an aperture; Quarterly of Applied Mathematics, 11, 77-86 (1953). Also EM 32, New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research.

[4] Bouwkamp, C.J. - On the freely vibrating circular disc and diffraction by circular discs and apertures, Physica, 16, 1-16 (1950).

[5] Meixner, J. - Die Kantenbedingung in der Theorie der Beugung elektromagnetischer Wellen an vollkommen leitenden Schirmen; Annalen der Physik (Ser. 6), 6, 1-7 (1949).

[6] Erdelyi, A. et al. - Higher transcendental functions; McGraw-Hill, New York, 1954, Vol. 2.

[7] Courant, R. and Hilbert, D. - Methods of mathematical physics; Interscience, New York, 1954, Vol. 1.

REPORTS AND PUBLICATIONS

The following is a complete list of reports and publications to date submitted under Contract No. AF 19(122)-42, Air Force Cambridge Research Center.

Unless otherwise indicated, reports are available to interested scientists on request.

* Copies not available for distribution.

** Reprints only available for distribution.

*EM-11	W. Sollfrey	The Variational Solution of Scattering Problems, March, 1949.
*EM-12	A. Leitner	Notes on Diffraction by a Circular Disk. April, 1949. J.Acoust. Soc. Amer., 21, 331 (1949).
*EM-13	J.B. Keller and H.B. Keller	Determination of Reflected and Transmitted Fields by Geometrical Optics, May, 1949. J. Opt. Soc. Amer., 40, 48 (1950).
*EM-14	R.K. Luneberg	Asymptotic Development of Steady-State Electromagnetic Fields, July, 1949.
*EM-15	R.K. Luneberg	Asymptotic Evaluations of Diffraction Integrals, October, 1949.
*EM-16	J.B. Keller and H.B. Keller	A Point Dipole in Spherically Symmetric Media, Feb., 1950.
*EM-17	B. Friedman	Numerical Methods for Evaluation of the Integrals for Virtual Height, February, 1950. L. Kraus, J.Geophys.Res., 58, 551 (1953).
**EM-18	J. Shmoys	Diffraction of Electromagnetic Waves by a Plane Wire Grating, March, 1950. J. Opt. Soc. Amer., 41, 324 (1951).
EM-19	A. Leitner	Effect of a Circular Groundplane on Antenna Radiation, April, 1950. J. Appl. Phys., 21, 1001 (1950).
*EM-20	J.B. Keller and S. Preiser	Determination of Reflected and Transmitted Fields by Geometrical Optics, Part II, April, 1950.
*EM-21	J.B. Keller and A. Blank	Diffraction and Reflection of Pulses by Wedges and Corners, June, 1950. Comm.Pure Appl.Math., 4, 75 (1951).
EM-22	V. Twersky	On the Scattered Reflection of Scalar Waves from Absorbent Surfaces, August, 1950. J. Acoust. Soc. Amer., 23, 336 (1951).
*EM-23	S. Karp and W. Sollfrey	Diffraction by a Dielectric Wedge with Application to Propagation through a Cold Front, October, 1950.

*EM-24	M. Kline	An Asymptotic Solution of Maxwell's Equations. Nov., 1950. Comm. Pure Appl. Math., 4, 225 (1951).
EM-25	S. Karp	Separation of Variables and Wiener-Hopf Techniques, Dec., 1950. Comm. Pure Appl. Math., 3, 411 (1950).
*EM-26	V. Twersky	On the Scattered Reflection of Electromagnetic Waves, Jan., 1951. J. Appl. Phys., 22, 825 (1951).
**EM-27	J. Shmoys	On the Definition of Virtual Height, February, 1951. J. Geophys. Res., 57, 95 (1952).
*EM-28	B. Friedman	The Dipole Field in an Inhomogeneous Atmosphere, March, 1951. Comm. Pure Appl. Math., 4, 317 (1951).
*EM-29	N. Marcuvitz	Field Representations in Spherically Stratified Regions, April, 1951. Comm. Pure Appl. Math., 4, 263 (1951).
*EM-30	B. Friedman	Report on a Conference on Dynamics of Ionized Media, May, 1951.
*EM-31	J. Lurye	Electromagnetic Scattering Matrices of Stratified Anisotropic Media, May, 1951.
EM-32	W. Magnus	Infinite Matrices Associated with Diffraction by an Aperture, May, 1951. Quart. Appl. Math. 11, 77, (1953).
EM-33	H.B. Keller and J.B. Keller	On Systems of Linear Ordinary Differential Equations, July, 1951.
*EM-34	V. Twersky	Multiple Scattering of Radiation, Part I: Arbitrary Configuration of Parallel Cylinders, Planar Configuration, Two Cylinders, July, 1951. J. Acoust. Soc. Amer., 24, 42 (1952); J. Appl. Phys., 23, 407 (1952).
*EM-35	S.N. Karp	The Natural Charge Distribution and Capacitance of a Finite Conical Shell, Sept., 1951. Accepted by Quart. Appl. Math.
EM-36	J.B. Keller	Parallel Reflection of Light by Plane Mirrors, October, 1951. Quart. Appl. Math., 11, 216 (1953).
*EM-37		The Theory of Electromagnetic Waves - A Symposium, Nov., 1951. Interscience Publishers, Inc., New York, December, 1951.
EM-38	A. Russek	Scattering Matrices for Ionosphere Models, December, 1951.
**EM-39	V. Twersky	Multiple Scattering of Radiation, Part II; The Grating, Dec., 1951. J. Appl. Phys., 23, 1099 (1952); J. Opt. Soc., 42, 855 (1952).
*EM-40	W. Magnus	On the Scattering Effect of a Rough Plane Surface, Jan., 1952.

EM-41	W. Sollfrey and J. Shmoys	Diffraction of Electromagnetic Waves by a Plane Wire Grating, II, February, 1952.
*EM-42	R.S. Phillips	Linear Ordinary Differential Operators of the Second Order, April, 1952.
**EM-43	I. Kay	Diffraction of an Arbitrary Pulse by a Wedge, April, 1952. Comm. Pure Appl. Math., 4, 419 (1953); Appendix by J.B. Keller, J. Appl. Phys., 23, 1267 (1952).
**EM-44	B. Friedman and J. Russek	Addition Theorems for Spherical Waves, June, 1952. Quart. Appl. Math., 12, 13 (1954).
*EM-45	W. Sollfrey	Diffraction of Pulses by Conducting Wedges and Cones, July, 1952.
*EM-46	J. Bazer and S.N. Karp	Propagation of Plane Electromagnetic Waves Past a Shoreline, July, 1952.
*EM-47	B. Friedman	Techniques of Applied Mathematics - Theory of Distributions, October, 1952.
*EM-48	M. Kline	An Asymptotic Solution of Linear Second-Order Hyperbolic Differential Equations, December, 1952. J. Rational Mechanics 3, 315 (1954).
*EM-49	J. Russek and V. Twersky	Graphs of the Function $E(N,\delta) = \sum_{n=1}^{N} n^{-1/2} e^{in\delta}$, February, 1953.
EM-50	C.J. Bouwkamp	Diffraction Theory - A Critique of Some Recent Developments, April, 1953. Reports on Progress in Phys., 17, 35 (1954).
*EM-51	J. Shmoys	Low-Frequency Propagation in an Exponential Ionospheric Layer, April, 1953.
*EM-52	S.N. Karp	Diffraction by a Tipped Wedge - with Applications to Blunt Edges, May, 1953.
EM-53	I. Kay	Diffraction of Pulses by Parabolic Cylinders and Paraboloid of Revolution, June, 1953.
**EM-54	V. Twersky	Certain Reflection and Transmission Coefficients, June, 1953. J. Appl. Phys., 25, 859 (1954); J.Appl.Phys., 24, 659 (1953).
EM-55	I. Kay and J.B. Keller	Asymptotic Evaluation of the Fields at a Caustic, August, 1953. J. Appl. Phys., 25, 876 (1954).
EM-56	H.B. Keller	Ionospheric Propagation of Plane Waves, August, 1953.
EM-57	H.B. Keller	On the Electromagnetic Field Equations in the Ionosphere, September, 1953.

EM-58	V. Twersky	Multiple Scattering of Waves by Planar Random Distributions of Parallel Cylinders and Bosses, October, 1953.
EM-59	V. Twersky	Multiple Scattering of Waves by a Volume Distribution of Parallel Cylinders, October, 1953.
EM-60	B. Friedman	Techniques of Applied Mathematics - Ordinary Differential Equations and Green's Functions, November, 1953.
EM-61	F. Reiche	On Diffraction by an Infinite Grating, December, 1953.
EM-62	C. Müller	Radiation Patterns and Radiation Fields. March, 1954. <u>J. Rational Mechanics</u>, <u>4</u>, 235, March, 1955.
EM-63	J. Shmoys	Propagation in Semi-Infinite Waveguides - Translations of Six Papers by L.A. Vajnshtejn, January, 1954.
EM-64	F.G. Friedlander	Diffraction of Pulses by a Circular Cylinder, April, 1954. <u>Comm. Pure Appl. Math.</u>, <u>7</u>, 705 (1954).
EM-65	J. Lurye	The Electromagnetic Field of a Dipole over a Dielectric Slab on a Finitely Conducting Ground Plane, July, 1954.
EM-66	J. Bazer and S.N. Karp	Potential Flow Through a Conical Pipe With an Application to Diffraction Theory, August, 1954.
EM-67	F.G. Friedlander and J.B. Keller	Asymptotic Expansions of Solutions of $(\nabla^2 + k^2)u = 0$, Sept., 1954.
EM-68	J. Meixner	Diffraction of Electromagnetic Waves by a Slit in a Conducting Plane Between Different Media, October, 1954.
EM-69	N. Marcuvitz	Field Representations in General Cylindrical Regions. I, November, 1954.
EM-70	W.E. Williams	Reflection and Refraction of Electromagnetic Waves at Plane Interfaces Between Dielectrics, November, 1954.
EM-71	S.N. Karp	The Effect of Discontinuities of Dielectric Constant on Fields near Singularities of Conductors, December, 1954.
EM-72	J. Meixner	The Behavior of Electromagnetic Fields at Edges, Dec., 1954.
EM-73	N. Chako	On Integral Relations Involving Products of Spheroidal Functions, January, 1955.
EM-74	I. Kay	The Inverse Scattering Problem, February, 1955.
EM-75	S.N. Karp and A. Russek	Diffraction by a Wide Slit, February, 1955.
EM-76	F.G. Friedlander	Propagation of a Pulse in an Inhomogeneous Medium, March, 1955.
EM-77	W.E. Williams	Step Discontinuities in Waveguides, April, 1955.
EM-78	D.S. Jones	A Critique of the Variational Method in Scattering Problems, May, 1955.

EM-80

Magnus

An infinite system of
linear equations arising
in diffraction theory.

EM-80

Magnus

An infinite system of linear
equations arising in diffrac-
tion theory.

Date Due

PRINTED IN U. S. A.